Valuation In Discrete And Continuous Time

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In this white paper we will build a model to calculate enterprise value in both discrete and continuous time.

Our Hypothetical Problem

We are tasked with calculating the enterprise value of a company given the following prameters...

Table 1: Model Assumptions

Symbol	Description	Value
C_0	Annualized cash flow at time zero	1,000,000
g	Annualized discrete time cash flow growth rate (%)	5.00
r	Annualized discrete time discount rate (%)	15.00

Given that the variable m represents the number of compounding periods in one year, build a valuation model to answer the following questions:

Question 1: What is enterprise value at time zero when m = 1?
Question 2: What is enterprise value at time zero when m = 12?
Question 3: What is enterprise value at time zero when m = infinity?

Discrete Time Valuation Equations

We will define the variable μ to be the cash flow growth rate. Given that we defined the variable g to be the discrete time annualized cash flow growth rate and the variable m to be the number of periods in one year the equation for the cash flow growth rate is...

$$\text{if...}\ \left(1+\frac{\mu}{m}\right)^m = 1+g\ \text{ ...then... }\ \mu = m\left[\left(1+g\right)^{\frac{1}{m}} - 1\right] \tag{1}$$

We will define the variable κ to be the risk-adjusted discount rate. Given that the variable r is the discrete time annualized discount rate and the variable m is the number of compounding periods in one year the equation for the risk-adjusted discount rate is...

$$\text{if...}\ \left(1+\frac{\kappa}{m}\right)^m = 1+r\ \text{ ...then... }\ \kappa = m\left[\left(1+r\right)^{\frac{1}{m}} - 1\right] \tag{2}$$

We will define the variable V_0 to be company value at time zero. Using Equations (1) and (2) above and the model parameters in Table 1 above the equation for company value in discrete time is...

$$\begin{aligned} V_0 &= \sum_{n=1}^{\infty}\frac{1}{m}\,C_0\left(1+\frac{\mu}{m}\right)^n\left(1+\frac{\kappa}{m}\right)^{-n} \\ &= \frac{1}{m}\,C_0\sum_{n=1}^{\infty}\left(\frac{1+\frac{\mu}{m}}{1+\frac{\kappa}{m}}\right)^n \\ &= \frac{1}{m}\,C_0\sum_{n=1}^{\infty}\theta^n\ \text{ ...where... }\ \theta = \frac{1+\frac{\mu}{m}}{1+\frac{\kappa}{m}} \\ &= \frac{1}{m}\,C_0\left[\sum_{n=0}^{\infty}\theta^n - 1\right] \end{aligned} \tag{3}$$

Note the solution to the following mathematical series...

$$\lim_{n\to\infty}\sum_{n=0}^{\infty}\theta^n=\frac{1}{1-\theta}\;\text{...when...}\;0<\theta<1 \tag{4}$$

Using Equation (4) above we can rewrite Equation (3) above as...

$$V_0=\frac{1}{m}\,C_0\left[\frac{1}{1-\theta}-1\right]=\frac{1}{m}\,C_0\,\frac{\theta}{1-\theta} \tag{5}$$

Using Appendix Equation (20) we can rewrite Equation (5) above as...

$$V_0=\frac{1}{m}\,C_0\left(1+\frac{\mu}{m}\right)\frac{1}{\frac{\kappa}{m}-\frac{\mu}{m}}=C_0\left(1+\frac{\mu}{m}\right)\Big/\left(\kappa-\mu\right) \tag{6}$$

Using Equations (1), (2) and (6) the equation for company value in discrete time is...

$$V_0=C_0\left(1+\frac{\mu}{m}\right)\Big/\left(\kappa-\mu\right)\;\text{...where...}\;\mu=m\left[\left(1+g\right)^{\frac{1}{m}}-1\right]\;\text{...and...}\;\kappa=m\left[\left(1+r\right)^{\frac{1}{m}}-1\right] \tag{7}$$

Continuous Time Valuation Equations

Equation (3) above is a discrete time equation for company value. The continuous time version of that equation is the limit of Equation (3) above as the number of annual periods (m) goes to infinity. The continuous time equation for company value is therefore...

$$V_0=\lim_{m\to\infty}\sum_{n=1}^{\infty}\frac{1}{m}\,C_0\left(1+\frac{\mu}{m}\right)^n\left(1+\frac{\kappa}{m}\right)^{-n} \tag{8}$$

Note that we can rewrite Equation (8) above as...

$$V_0=\lim_{m\to\infty}\sum_{n=1}^{\infty}\frac{1}{m}\,C_0\left\{\left(1+\frac{\mu}{m}\right)^m\right\}^{\frac{n}{m}}\left\{\left(1+\frac{\kappa}{m}\right)^m\right\}^{-\frac{n}{m}} \tag{9}$$

We will define the variable t to be time in years. Using Equation (9) above we have the following limits as the number of annual periods goes to infinity...

$$\lim_{m\to\infty}\frac{1}{m}=\delta t\;\Bigg|\;\lim_{m\to\infty}\left(1+\frac{\mu}{m}\right)^m=\text{Exp}\left\{\mu\right\}\;\Bigg|\;\lim_{m\to\infty}\left(1+\frac{\kappa}{m}\right)^m=\text{Exp}\left\{\kappa\right\}\;\Bigg|\;\frac{n}{m}=t \tag{10}$$

Using Equation (10) above we can rewrite Equation (9) above as...

$$V_0=\int_0^{\infty}C_0\left(\text{Exp}\left\{\mu\right\}\right)^t\left(\text{Exp}\left\{\kappa\right\}\right)^{-t}\delta t=C_0\int_0^{\infty}\text{Exp}\left\{\mu\,t\right\}\text{Exp}\left\{-\kappa\,t\right\}\delta t \tag{11}$$

Using Appendix Equation (22) below the solution to Equation (11) above is...

$$V_0=\frac{C_0}{\kappa-\mu} \tag{12}$$

Given that we defined the variable g to be the discrete time annualized cash flow growth rate the equation for the cash flow growth rate in continuous time is...

$$\text{if...}\;\text{Exp}\left\{\mu\right\}=1+g\;\text{...then...}\;\mu=\ln\left(1+g\right) \tag{13}$$

Given that the variable r is the discrete time annualized discount rate and the variable m is the number of compounding periods in one year the equation for the risk-adjusted discount rate in continuous time is...

$$\text{if...}\;\text{Exp}\left\{\kappa\right\}=1+r\;\text{...then...}\;\kappa=\ln\left(1+r\right) \tag{14}$$

Using Equations (12), (13) and (14) above the equation for company value in continuous time is...

$$V_0=\frac{C_0}{\kappa-\mu}\;\text{...where...}\;\mu=\ln\left(1+g\right)\;\text{...and...}\;\kappa=\ln\left(1+r\right) \tag{15}$$

The Answers To Our Hypothetical Problem

Question 1: What is enterprise value at time zero when m = 1?

Using Equation (7) above the answer to this question is...

$$V_0 = 1,000,000 \times \left(1 + \frac{0.05000}{1}\right) \Big/ \left(0.15000 - 0.05000\right) = 10,500,000$$
$$\text{...where... } \mu = 1 \times \left[\left(1 + 0.05\right)^{\frac{1}{1}} - 1\right] = 0.05000 \text{ ...and... } \kappa = 1 \times \left[\left(1 + 0.15\right)^{\frac{1}{1}} - 1\right] = 0.15000 \tag{16}$$

Question 2: What is enterprise value at time zero when m = 12?

Using Equation (7) above the answer to this question is...

$$V_0 = 1,000,000 \times \left(1 + \frac{0.04889}{12}\right) \Big/ \left(0.14058 - 0.04889\right) = 10,951,000$$
$$\text{...where... } \mu = 12 \times \left[\left(1 + 0.05\right)^{\frac{1}{12}} - 1\right] = 0.04889 \text{ ...and... } \kappa = 12 \times \left[\left(1 + 0.15\right)^{\frac{1}{12}} - 1\right] = 0.14058 \tag{17}$$

Question 3: What is enterprise value at time zero when m = infinity?

Using Equation (15) above the answer to this question is...

$$V_0 = \frac{1,000,000}{0.13976 - 0.04879} = 10,992,000$$
$$\text{...where... } \mu = \ln\left(1 + 0.05000\right) = 0.04879 \text{ ...and... } \kappa = \ln\left(1 + 0.15000\right) = 0.13976 \tag{18}$$

Appendix

A. We want to solve the following equation...

$$x = \frac{\theta}{1 - \theta} \text{ ...when... } \theta = \frac{1 + a}{1 + b} \tag{19}$$

The solution to Equation (19) above is...

$$x = \frac{\frac{1+a}{1+b}}{1 - \frac{1+a}{1+b}} = \frac{\frac{1+a}{1+b}}{\frac{1+b-1-a}{1+b}} = \frac{1 + a}{b - a} \tag{20}$$

B. We want to solve the following equation...

$$\int_0^\infty \text{Exp}\left\{a\,t\right\} \text{Exp}\left\{-\,b\,t\right\} \delta t = \int_0^\infty \text{Exp}\left\{(a - b)\,t\right\} \delta t = \frac{1}{a - b} \text{Exp}\left\{(a - b)\,t\right\} \Big[_0^\infty \tag{21}$$

If $a - b$ is less than zero then the solution to Equation (21) above is...

$$\int_0^\infty \text{Exp}\left\{a\,t\right\} \text{Exp}\left\{-\,b\,t\right\} \delta t = \frac{1}{a - b}\left(0 - 1\right) = \frac{1}{b - a} \tag{22}$$